

A new kind of functional differential equations

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Abstract

In this paper we introduce and investigate a new kind of functional (including ordinary and evolutionary partial) differential equations. The main goal of this paper is to explore our new philosophy by some examples on functional ODEs and PDEs. For some typical examples, we prove the global existence of smooth solutions, analyze some interesting properties enjoyed by these solutions, and illustrate the differences between this new class of equations and the traditional ones. This kind of functional differential equations is a new and powerful tool to study some problems arising from both mathematics and physics, more applications in particular to differential geometry and fundamental physics can be expected.

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1 Introduction

A differential equation is an equation involving an unknown function of one or two or more variables and certain its (partial) derivatives. In general, we can write out symbolically a typical differential equation, as follows. Fix an integer $k \geq 1$ and let Ω denote an open subset of \mathbb{R}^n ($n \geq 0$). An expression of the form

$$\mathcal{P}(D^k u(t, x), D^{k-1} u(t, x), \dots, Du(t, x), u(t, x), t, x) = 0 \quad (x = (x_1 \cdots, x_n) \in \Omega) \quad (1.1)$$

is called a k -th order differential equation, where

$$\mathcal{P} : \mathbb{R}^{(n+1)^k} \times \mathbb{R}^{(n+1)^{k-1}} \times \dots \times \mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$$

is given and $u = u(t, x_1 \cdots, x_n)$ is the unknown. The variable t always denotes time. Obviously, if $n = 0$, then (1.1) is an ordinary differential equation; if $n \geq 1$, then it is partial differential equations.

Instead of (1.1), we are interested in the following functional differential equation

$$\mathcal{F}(D^k u(t, x), D^k u(t/2, x), D^{k-1} u(t, x), D^{k-1} u(t/2, x), \dots, Du(t, x), Du(t/2, x), u(t, x), u(t/2, x), t, x) = 0, \quad (1.2)$$

where

$$\mathcal{F} : \mathbb{R}^{(n+1)^k} \times \mathbb{R}^{(n+1)^k} \times \mathbb{R}^{(n+1)^{k-1}} \times \mathbb{R}^{(n+1)^{k-1}} \times \dots \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$$

is given and $u = u(t, x_1 \cdots, x_n)$ is the unknown. The equation (1.2) is also known as “nonlocal”, some time called *time-delay*¹, differential equations. This kind of new time delay effect can also be introduced in the study of some problems arising in fundamental physics in a way similar to the work [1]. In this paper, we shall explore our new philosophy by some simple but interesting examples.

In fact, $t/2$ in (1.2) can be replaced by t/α ($\alpha > 1$) or $\phi(t)$ with $0 < \phi(t) < t$. In this case, we can develop a similar theory.

Some typical examples are as follows:

1.1 Functional ODEs

- Linear-like equation

$$\frac{dx(t)}{dt} = x\left(\frac{t}{2}\right). \quad (1.3)$$

¹Usually, the traditional time-delay equation means that the time variable in the equation is $t - r$ for some constant r instead of $t/2$ in this paper (see [17]).

- Riccati-like equation

$$\frac{dx(t)}{dt} = x^2\left(\frac{t}{2}\right). \quad (1.4)$$

- A system of functional ODEs

$$\frac{dx(t)}{dt} = A(t)x(t) + F(x(\phi(t))), \quad t \geq 0, \quad (1.5)$$

where $x(t) = (x_1(t), \dots, x_n(t))^T$, $A(t)$ is a continuous $n \times n$ matrix-valued function of $t \in \mathbb{R}^+$, $F = (F_1, \dots, F_n)^T$ is a locally Lipschitzian continuous vector-valued function of $x \in \mathbb{R}^n$, and ϕ is a continuous real-valued function of $t \in \mathbb{R}^+$ with

$$0 < \phi(t) < t \quad \text{for all } t > 0. \quad (1.6)$$

1.2 Functional PDEs

- Burgers-like equation

$$u_t(t, x) + u(t/2, x)u_x(t, x) = 0. \quad (1.7)$$

- Heat-like equation

$$u_t(t, x) - u_{xx}(t/2, x) = 0. \quad (1.8)$$

- Wave-like equation

$$u_{tt}(t, x) - u_{xx}(t/2, x) = 0. \quad (1.9)$$

1.3 Functional geometric flows

- Functional Ricci flow

Let \mathcal{M} be an n -dimensional complete Riemannian manifold with Riemannian metric g_{ij} , the Hamilton's Ricci flow is given by the evolution equation (see [6])

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \quad (1.10)$$

for a family of Riemannian metrics $g_{ij}(t)$ on \mathcal{M} . (1.10) is a nonlinear system of second order partial differential equations on the metric g_{ij} . The functional Ricci flow under consideration reads

$$\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}|_{\frac{t}{2}}, \quad (1.11)$$

where $\bullet|_{\frac{t}{2}}$ stands for the value of \bullet at the time $\frac{t}{2}$. Thus, in (1.11), $R_{ij}|_{\frac{t}{2}}$ is the Ricci curvature corresponding to the metric $g_{ij}(t/2)$, i.e., the metric at the time $\frac{t}{2}$. Obviously, (1.11) is a system

of functional partial differential equations. Noting the special time-delay effect, we may consider the Cauchy problem for (1.11) with the initial data

$$t = 0 : \quad g_{ij} = g_{ij}^0, \quad (1.12)$$

where g_{ij}^0 is a given Riemannian metric on \mathcal{M} . Therefore, the Cauchy problem (1.11)-(1.12) gives the evolution of the metric g_{ij}^0 under the flow (1.11).

- Functional mean curvature flow

Let \mathcal{M} be an n -dimensional smooth manifold and

$$X(\cdot, t) : \quad \mathcal{M} \rightarrow \mathbb{R}^{n+1}$$

be a one-parameter family of smooth hypersurface immersions in \mathbb{R}^{n+1} . The traditional mean curvature flow is defined by (see [2]-[5] and [8])

$$\frac{\partial}{\partial t} X(x, t) = H(x, t) \vec{n}(x, t), \quad \forall x \in \mathcal{M}, \quad \forall t \geq 0, \quad (1.13)$$

where $H(x, t)$ is the mean curvature of $X(x, t)$ and $\vec{n}(x, t)$ is the unit inner normal vector on $X(\cdot, t)$. The nonlocal mean curvature flow under consideration in the present paper is

$$\frac{\partial}{\partial t} X(x, t) = H(x, t/2) \vec{n}(x, t/2), \quad \forall x \in \mathcal{M}, \quad \forall t \geq 0, \quad (1.14)$$

It is easy to see that (1.14) is a system of nonlocal partial differential equations. Noting the special time-delay effect, we may consider the Cauchy problem for (1.14) with the initial data

$$t = 0 : \quad X = X_0, \quad (1.15)$$

where X_0 is a given hypersurface. Thus, the Cauchy problem (1.15) gives the evolution of the hypersurface X_0 under the flow (1.14).

Remark 1.1 *For the inverse mean curvature flow (see [9]) and the hyperbolic mean curvature flow (see [7], [12] and [15]), we have a similar discussion.*

- Functional hyperbolic geometric flow

Let \mathcal{M} be an n -dimensional complete Riemannian manifold with Riemannian metric g_{ij} , the hyperbolic geometric flow considered in [10]-[11] and [13] is described by the evolution equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} \quad (1.16)$$

for a family of Riemannian metrics $g_{ij}(t)$ on \mathcal{M} . (1.16) is a nonlinear system of second order partial differential equations on the metric g_{ij} . The functional hyperbolic geometric flow under considered here reads

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2 R_{ij}|_{\frac{t}{2}}. \quad (1.17)$$

Obviously, (1.17) is a system of functional partial differential equations. Noting the special time-delay effect, we may consider the Cauchy problem for (1.17) with the initial data

$$t = 0 : \quad g_{ij} = g_{ij}^0, \quad \frac{\partial g_{ij}}{\partial t} = k_{ij}^0, \quad (1.18)$$

where g_{ij}^0 is a given Riemannian metric on \mathcal{M} , and k_{ij}^0 is a symmetric tensor on \mathcal{M} . Therefore, the Cauchy problem (1.17)-(1.18) gives the evolution of the metric g_{ij}^0 under the flow (1.17).

1.4 A shifted view of fundamental physics

Atiyah and Moore speculated on the role of relativistic versions of delayed differential equations in fundamental physics (see [1]). Since relativistic invariance implies that one must consider both advanced and retarded terms in the equations, in [1] they refereed to them as shifted equations and showed that the shifted Dirac equation has some novel properties and a tentative formulation of shifted Einstein-Maxwell equations naturally incorporates a small but nonzero cosmological constant.

1.5 Aim and outline of the paper

The main goal of this paper is to explore our new philosophy by some examples on functional ODEs and PDEs mentioned above. For typical examples² (e.g., (1.3) and (1.8)), we prove the global existence of smooth solutions, analyze some interesting properties enjoyed by these solutions, and illustrate the differences between this new class of equations and the traditional ones.

The paper is organized as follows. In Section 2, we explore our philosophy on this new kind of functional ODEs and PDEs under consideration. Sections 3 and 4 are devoted to investigating the equation (1.3) and presenting some interesting properties enjoyed by the solution of (1.3), while Section 5 is devoted to studying the equation (1.8).

²In fact, for the hyperbolic equations (1.7) and (1.9), we can prove the global existence of smooth solutions in a manner similar to [16].

2 New philosophy

In this section, we explore our new philosophy and the differences between functional differential equations and traditional ones by some examples.

Example 2.1 Taking the equation (1.4) as the first example, we consider the Cauchy problem for (1.4), i.e.,

$$\frac{dx(t)}{dt} = x^2\left(\frac{t}{2}\right)$$

with the initial data

$$x(0) = 1. \quad (2.1)$$

It is easy to see that the Cauchy problem (1.4), (2.1) has the following global smooth solution

$$x(t) = e^t \quad \text{for all } t. \quad (2.2)$$

On the other hand, the solution of the Cauchy problem for the traditional Riccati equation

$$\frac{dx(t)}{dt} = x^2(t) \quad (2.3)$$

with the initial data (2.1) reads

$$x(t) = \frac{1}{1-t}. \quad (2.4)$$

Clearly, the solution blows up at the time $t = 1$, since

$$x(t) \nearrow \infty \quad \text{as} \quad t \nearrow 1. \quad (2.5)$$

Example 2.2 We next consider another simple but interesting example — the equation (1.3). To illustrate the difference between the equation (1.3) and the traditional linear ODE, we first recall the solution of the following Cauchy problem

$$\begin{cases} \frac{dx(t)}{dt} = x(t), \\ x(0) = 1. \end{cases} \quad (2.6)$$

It is well-known that the solution can be explicitly expressed by

$$x(t) = e^t \quad \text{for all } t. \quad (2.7)$$

Obviously, the solution is a strictly increasing function for t . However, for the following Cauchy problem for the equation (1.3)

$$\begin{cases} \frac{dx(t)}{dt} = x\left(\frac{t}{2}\right), \\ x(0) = 1, \end{cases} \quad (2.8)$$

in next section, we will prove

Theorem 2.1 *The Cauchy problem (2.8) admits the global solution shown as in the following figures:*

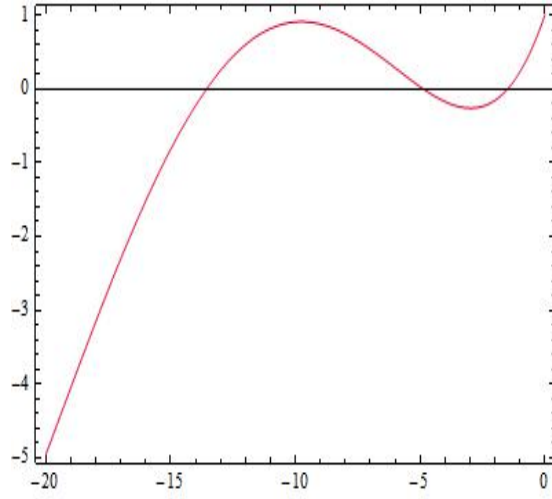


Figure 1: The figure of the solution on the interval $[-20, 0]$

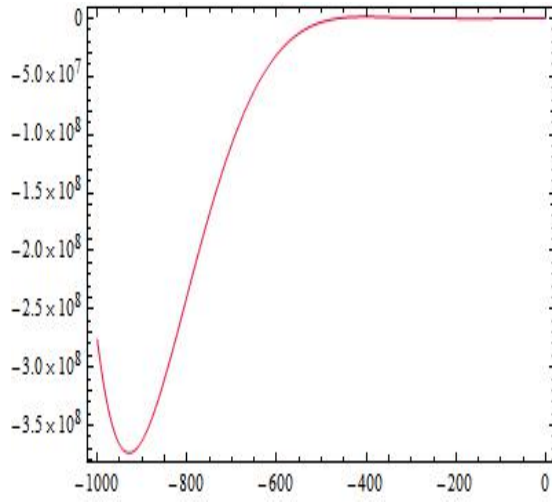


Figure 2: The figure of the solution on the interval $[-1000, 0]$

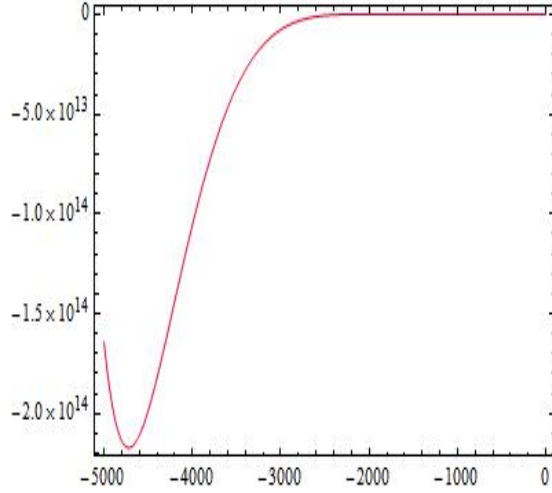


Figure 3: The figure of the solution on the interval $[-5000, 0]$

On the other hand, on the interval $[0, \infty)$, the solution is strictly increasing and goes to infinity as t tends to infinity.

For more interesting properties of the solution of the Cauchy problem (2.8), see next section.

Remark 2.1 (2.7) and Theorem 2.1 give the essential differences between (2.8) and the traditional linear equation in (2.6).

In fact, more generally, we consider the Cauchy problem for a system of functional differential equations

$$\begin{cases} \frac{dx(t)}{dt} = A(t)x(t) + F(x(\phi(t)))x(t) + G(x(\psi(t))), \\ x(0) = x_0, \end{cases} \quad (2.9)$$

where $x(t) = (x_1(t), \dots, x_n(t))^T$ is the unknown vector-valued function, x_0 is a constant vector. We have (see [16])

Theorem 2.2 Let $A(t)$ be a continuous $n \times n$ matrix-valued function of $t \in \mathbb{R}^+$, F, G be locally Lipschitzian continuous vector-valued functions of $x \in \mathbb{R}^n$, and ϕ, ψ be continuous real-valued functions of $t \in \mathbb{R}^+$ with

$$0 < \phi(t), \psi(t) < t \quad \text{for all } t > 0.$$

Then Cauchy problem (2.9) admits a unique global C^1 solution $x = x(t)$ on $t \geq 0$.

Example 2.3 *We now consider the following Cauchy problem for the Burgers equation*

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ t = 0 : u = \sin x. \end{cases} \quad (2.10)$$

It is well-known that the smooth solution of the Cauchy problem (2.10) only exists on the strip $[0, 1) \times \mathbb{R}$, and singularities, i.e., shock waves will appear at the time $t = 1$ (e.g., at the point $(1, 0)$). However, the Cauchy problem for the Burgers-like equation (1.7)

$$\begin{cases} u_t(t, x) + u(t/2, x)u_x(t, x) = 0, \\ t = 0 : u = \sin x. \end{cases} \quad (2.11)$$

always admits a global smooth solution on the domain $\mathbb{R}^+ \times \mathbb{R}$. See [14]. In fact, in the work [14], the authors generalize the above results to the case of quasilinear hyperbolic systems of partial differential equations.

Clearly, Examples 2.1-2.3 give new philosophy enjoyed by functional differential equations (FDEs) and the essential differences between FDEs and traditional ones.

3 Functional ordinary differential equation: Main results

In this section, we investigate solutions to functional differential equations mentioned above. For simplicity, in this section we only consider the solution of the following Cauchy problem

$$\begin{cases} \frac{dy(x)}{dx} = y\left(\frac{x}{2}\right), \\ y(0) = 1. \end{cases} \quad (3.1)$$

We shall prove the global existence of smooth solution, show regularities of zeros distribution and derive an exact estimate of the oscillation amplitude. In fact, our results can be easily generalized to other functional differential equations with a similar form.

Throughout this section, we denote the solution of the Cauchy problem (3.1) by $y = f(x)$. Direct calculation shows that the solution can be represented in power series form

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n! 2^{\frac{n(n-1)}{2}}}. \quad (3.2)$$

In order to investigate the properties of the solution, we first prove the following useful lemma on the truncated polynomials of $f(x)$.

Lemma 3.1 *For any integer $N > 2$, the truncated polynomial of $f(x)$, i.e.,*

$$P_N(x) = \sum_{n=0}^N \frac{x^n}{n! 2^{\frac{n(n-1)}{2}}} \quad (3.3)$$

has $N-2$ real zeros denoted by

$$\{r_n\}_{1 \leq n \leq N-2} \quad \text{with} \quad r_{n+1} < r_n < 0, \quad (3.4)$$

and a pair of imaginary zeros $z_N, \overline{z_N}$. Moreover, the following estimates hold

$$|r_n| < (n+2)2^n \quad (1 \leq n \leq N-2) \quad (3.5)$$

and

$$|z_N|^2 > 2^N. \quad (3.6)$$

By Lemma 3.1, we shall prove the following theorem on regularities of zeros distribution of the solution in Section 4.

Theorem 3.1 *Let \mathbb{A} be the set of zeros of $f(z)$ in \mathbb{C} . Then $\mathbb{A} \subset \mathbb{R}$ is countably infinite, and it can be represented as $\{x_n\}_{n \geq 1}$ with the property*

$$x_{n+1} < x_n < 0.$$

Moreover, for any fixed integer $n \geq 1$, it holds that

$$x_n = -(n + \theta_n)2^{n-1} \quad (0 < \theta_n < 1). \quad (3.7)$$

In fact, in next section we can show the following more precise estimate on the parameter θ_n in (3.7).

Remark 3.1 *For large n , we have*

$$\frac{66}{25n} < \theta_n < \frac{2167}{789n}.$$

Moreover, it holds that

$$\lim_{n \rightarrow \infty} n\theta_n = C \approx 2.744. \quad (3.8)$$

Moreover, based on Theorem 3.1, we will prove the following theorem on the oscillation amplitude of the solution.

Theorem 3.2 *Let \mathbb{B} be the set of critical points of $y = f(x)$ in \mathbb{R} . Then*

$$\mathbb{B} = \{2x_1, 2x_2, \dots, 2x_n, \dots\},$$

the extrema of $y = f(x)$ can be estimated as

$$\log_2 |f(2x_n)| < \frac{1}{2}n^2 + 2n, \quad (3.9)$$

and their signs are alternate, i.e.,

$$(-1)^n f(2x_n) > 0. \quad (3.10)$$

Moreover, for $x < 0$ with large $|x|$, the following oscillation amplitude estimate holds

$$\log_2 |f(x)| < \frac{1}{2}(\log_2 |x|)^2 + 2\log_2 |x|. \quad (3.11)$$

Remark 3.2 *On the other hand, by Gronwall inequality, we can derive the following rough estimate on oscillation amplitude for $x < 0$*

$$\log_2 |f(x)| < |x| \log_2 e. \quad (3.12)$$

Since

$$|x| \gg (\log_2 |x|)^2$$

for large $|x|$, the estimate (3.11) is much better than (3.12).

4 Functional ordinary differential equation: Proof of Theorems 3.1 and 3.2

In this section, we prove Theorems 3.1 and 3.2. As before, we still denote the solution by $y = f(x)$ throughout this section.

Recall that $f(x)$ in power series form reads

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n! 2^{\frac{n(n-1)}{2}}}. \quad (4.1)$$

Proposition 4.1 *For integer $k \geq 1$, the signs of values $f(-k2^{k-1})$ and $f(-(k+1)2^{k-1})$ both are alternate, that is,*

$$(-1)^k f(-k2^{k-1}) < 0 \quad (4.2)$$

and

$$(-1)^k f(-(k+1)2^{k-1}) > 0. \quad (4.3)$$

Proof. We first consider the value of the solution $y = f(x)$ at the point $-k2^{k-1}$, namely, the value $f(-k2^{k-1})$.

Direct calculation leads to

$$f(-k2^{k-1}) = \sum_{n=0}^{\infty} \frac{k^n}{n!} 2^{nk-n(n+1)/2} (-1)^n \triangleq \sum_{n=0}^{\infty} (-1)^n u_n. \quad (4.4)$$

Then,

$$\frac{u_{n+1}}{u_n} = \frac{k}{n+1} 2^{k-n-1}. \quad (4.5)$$

Therefore,

$$u_n \text{ is increasing for } n < k; \quad \text{while } u_n \text{ is decreasing for } n > k. \quad (4.6)$$

It is easy to check that

$$u_{2k} = \frac{k^{2k}}{(2k)!} 2^{-k} < \frac{k^{2k}}{\left(\frac{2k}{e}\right)^{2k}} 2^{-k} = \left(\frac{e}{2\sqrt{2}}\right)^{2k} < 1. \quad (4.7)$$

Based on this, we denote

$$v_j = u_{2k-j-1} - u_j \quad (0 \leq j \leq k-1). \quad (4.8)$$

We have the following lemma which will be proved later.

Lemma 4.1 *The sequence $\{v_j\}_{0 \leq j \leq k-1}$ enjoys following properties*

$$v_{k-1} = 0, \quad v_j > 0 \quad (0 \leq j \leq k-2), \quad (4.9)$$

$$v_j < v_{j+1} \quad (0 \leq j \leq k-4) \quad (4.10)$$

and

$$v_{k-2} - v_{k-3} + v_{k-4} - v_{k-5} > 0. \quad (4.11)$$

On one hand, by Lemma 4.1 we obtain

$$(-1)^k \sum_{n=0}^{2k-1} u_n (-1)^n = (-v_{k-2} + v_{k-3} - v_{k-4} + v_{k-5}) - \sum_{j=0}^{k-6} v_j (-1)^{k-j} < 0. \quad (4.12)$$

On the other hand, direct calculation gives

$$v_0 = \frac{k^{2k-1}}{(2k-1)!} - 1 \geq 1 \quad (k \geq 4) \quad (4.13)$$

and

$$v_1 - v_0 = \left(\frac{k^{2k-2}}{(2k-2)!} - k\right) 2^{k-1} - \frac{k^{2k-1}}{(2k-1)!} + 1 \geq 1 \quad (k \geq 5). \quad (4.14)$$

Hence,

$$(-1)^k \sum_{n=0}^{2k-1} u_n (-1)^n < -1 \quad (k \geq 4). \quad (4.15)$$

Noting

$$\left| \sum_{n=2k}^{\infty} u_n (-1)^n \right| < u_{2k} < 1, \quad (4.16)$$

for $k \geq 4$ we have

$$(-1)^k \sum_{n=0}^{\infty} u_n (-1)^n < (-1)^k \sum_{n=0}^{2k-1} u_n (-1)^n + \left| \sum_{n=2k}^{\infty} u_n (-1)^n \right| < (-1)^k \sum_{n=0}^{2k-1} u_n (-1)^n + 1 < 0. \quad (4.17)$$

Similarly, it is easy to check that the desired inequalities hold for $k = 1, 2, 3$. This finishes the proof of the first part of Proposition 4.1.

We next show the second part of Proposition 4.1. In fact, the method of proof is the same as above, here we only state some essential differences.

Direct calculation yields

$$f(-(k+1)2^{k-1}) = \sum_{n=0}^{\infty} \frac{(k+1)^n}{n!} 2^{nk-n(n+1)/2} (-1)^n \triangleq \sum_{n=0}^{\infty} u_n (-1)^n. \quad (4.18)$$

Then,

$$\frac{u_{n+1}}{u_n} = \frac{k+1}{n+1} 2^{k-n-1}. \quad (4.19)$$

Therefore,

$$u_n \text{ is increasing for } n < k; \quad \text{while } u_n \text{ is decreasing for } n > k. \quad (4.20)$$

Noting

$$u_2 = 1, \quad u_{2k} = \frac{(k+1)^{2k}}{(2k)!} 2^{-k} < \frac{(k+1)^{2k}}{\sqrt{4\pi k} \left(\frac{2k}{e}\right)^{2k}} 2^{-k} = \frac{1}{\sqrt{4\pi k}} \left(\frac{e(k+1)}{2\sqrt{2k}}\right)^{2k} < 1 \quad (k > 1) \quad (4.21)$$

and denoting

$$v_j = u_{2k-j-1} - u_j \quad (0 \leq j \leq k-1), \quad (4.22)$$

in a manner similar to Lemma 4.1, we have the following lemma which will be proved later.

Lemma 4.2 *The sequence $\{v_j\}_{0 \leq j \leq k-1}$ enjoys following properties*

$$v_j > 0 \quad (0 \leq j \leq k-1), \quad (4.23)$$

$$v_j < v_{j+1} \quad (0 \leq j \leq k-3) \quad (4.24)$$

and

$$v_{k-1} - v_{k-2} + v_{k-3} - v_{k-4} > 0. \quad (4.25)$$

By Lemma 4.2, similar to (4.12), we have

$$(-1)^k \sum_{n=0}^{2k-1} u_n (-1)^n = (v_{k-1} - v_{k-2} + v_{k-3} - v_{k-4}) + \sum_{j=0}^{k-5} v_j (-1)^{k-j-1} > 0. \quad (4.26)$$

Direct calculation yields

$$v_0 = \frac{(k+1)^{2k-1}}{(2k-1)!} - 1 \geq 1 \quad (k \geq 1) \quad (4.27)$$

and

$$v_1 - v_0 = \left(\frac{(k+1)^{2k-2}}{(2k-2)!} - k - 1 \right) 2^{k-1} - \frac{(k+1)^{2k-1}}{(2k-1)!} + 1 \geq 1 \quad (k \geq 4). \quad (4.28)$$

Hence, it holds that

$$(-1)^k \sum_{n=0}^{2k-1} u_n (-1)^n > 1 \quad (k \geq 4). \quad (4.29)$$

Noting

$$\left| \sum_{n=2k}^{\infty} u_n (-1)^n \right| < u_{2k} \leq 1, \quad (4.30)$$

we obtain

$$(-1)^k \sum_{n=0}^{\infty} u_n (-1)^n > (-1)^k \sum_{n=0}^{2k-1} u_n (-1)^n - \left| \sum_{n=2k}^{\infty} u_n (-1)^n \right| > (-1)^k \sum_{n=0}^{2k-1} u_n (-1)^n - 1 > 0 \quad (k \geq 4). \quad (4.31)$$

Moreover, it is easy to check that the inequality also holds for $k = 1, 2, 3$. This proves the desired (4.3). Thus, Proposition 4.1 has been proved. \blacksquare

In what follows, we shall show Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. On one hand, we notice

$$v_{k-1} = u_k - u_{k-1} = 0. \quad (4.32)$$

On the other hand, we have

$$\prod_{i=1}^{2k-1-2j} (j+i) < k^{2k-1-2j} \quad (0 \leq j \leq k-2). \quad (4.33)$$

Hence,

$$\frac{k^j}{j!} < \frac{k^{2k-1-j}}{(2k-1-j)!} \quad (0 \leq j \leq k-2), \quad (4.34)$$

namely,

$$u_j < u_{2k-1-j} \quad (0 \leq j \leq k-2) \quad (4.35)$$

and

$$v_j > 0 \quad (0 \leq j \leq k-2). \quad (4.36)$$

This proves (4.9).

In order to prove (4.10), namely,

$$v_j < v_{j+1} \quad (0 \leq j \leq k-4), \quad (4.37)$$

by direct calculation it suffices to show

$$\left(2^{k-j-1} - \frac{j+1}{k}\right) w_j < 2^{k-j-1} - \frac{k}{2k-1-j} \quad (0 \leq j \leq k-4), \quad (4.38)$$

where

$$w_j = \frac{(2k-2-j)!}{(j+1)!k^{2k-3-2j}}. \quad (4.39)$$

Note that

$$\frac{w_{j+1}}{w_j} = \frac{k^2}{(j+2)(2k-2-j)} > 1. \quad (4.40)$$

This is to say, w_j is increasing. Hence, we only need to prove

$$\left(2^{k-j-1} - \frac{j+1}{k}\right) w_{k-4} < 2^{k-j-1} - \frac{k}{2k-1-j} \quad (0 \leq j \leq k-4), \quad (4.41)$$

i.e.,

$$\left(2^{k-j-1} - \frac{j+1}{k}\right) \frac{(k^2-1)(k^2-4)}{k^4} < 2^{k-j-1} - \frac{k}{2k-1-j} \quad (0 \leq j \leq k-4) \quad (4.42)$$

and

$$\frac{1}{2k-1-j} - \frac{(j+1)(k^2-1)(k^2-4)}{k^6} - 2^{k-j-1} \frac{5k^2-4}{k^5} < 0 \quad (0 \leq j \leq k-4). \quad (4.43)$$

Denote

$$t = 2k-1-j \quad (k+3 \leq t \leq 2k-1) \quad (4.44)$$

and

$$g(t) = \frac{1}{t} - \frac{(2k-t)(k^2-1)(k^2-4)}{k^6} - 2^{t-k} \frac{5k^2-4}{k^5}. \quad (4.45)$$

In what follows, we shall prove

$$g(t) < 0,$$

which will leads to our desired result immediately.

In fact, direct calculation yields

$$g'(t) = -\frac{1}{t^2} + \frac{(k^2-1)(k^2-4)}{k^6} - 2^{t-k} \frac{5k^2-4}{k^5} \ln 2 \quad (4.46)$$

and

$$g''(t) = \frac{2}{t^3} - 2^{t-k} \frac{5k^2-4}{k^5} (\ln 2)^2. \quad (4.47)$$

Combining

$$g''(k+3) = \frac{2}{(k+3)^3} - \frac{8(5k^2-4)}{k^5}(\ln 2)^2 = -\frac{40(\ln 2)^2-2}{k^3} + O(k^{-4}) < 0, \quad (4.48)$$

$$g'(k+3) = -\frac{1}{(k+3)^2} + \frac{(k^2-1)(k^2-4)}{k^6} - \frac{8(5k^2-4)}{k^5} \ln 2 = -\frac{40 \ln 2 - 6}{k^3} + O(k^{-4}) < 0 \quad (4.49)$$

and

$$g(k+3) = \frac{1}{k+3} - \frac{(k-3)(k^2-1)(k^2-4)}{k^6} - \frac{8(5k^2-4)}{k^5} = -\frac{26}{k^3} + O(k^{-4}) < 0 \quad (4.50)$$

gives

$$g(t) \leq g(k+3) < 0. \quad (4.51)$$

This gives the desired (4.10).

Finally, we prove (4.11).

Noting

$$v_{k-3} - v_{k-2} = u_{k+2} - u_{k-3} - u_{k+1} + u_{k-2} = \frac{k^{k-4}}{(k+4)!} (128k^6 + O(k^5)) 2^{\frac{k^2-k-20}{2}} \quad (4.52)$$

and

$$v_{k-4} - v_{k-5} = u_{k+3} - u_{k-4} - u_{k+4} + u_{k-5} = \frac{k^{k-4}}{(k+4)!} (194k^6 + O(k^5)) 2^{\frac{k^2-k-20}{2}}, \quad (4.53)$$

we obtain

$$v_{k-2} - v_{k-3} + v_{k-4} - v_{k-5} = \frac{k^{k-4}}{(k+4)!} (66k^6 + O(k^5)) 2^{\frac{k^2-k-20}{2}} > 0, \quad (4.54)$$

which is nothing but the desired estimate (4.11). Thus, the proof of Lemma 4.1 is completed.

■

We next prove Lemma 4.2.

Proof of Lemma 4.2. Noting

$$\prod_{i=1}^{2k-1-2j} (j+i) < (k+1)^{2k-1-2j} \quad (0 \leq j \leq k-1), \quad (4.55)$$

we have

$$\frac{(k+1)^j}{j!} < \frac{(k+1)^{2k-1-j}}{(2k-1-j)!} \quad (0 \leq j \leq k-1). \quad (4.56)$$

This implies that

$$u_j < u_{2k-1-j} \quad (0 \leq j \leq k-1), \quad (4.57)$$

and then

$$v_j > 0 \quad (0 \leq j \leq k-1). \quad (4.58)$$

This is nothing but the desired (4.23).

In order to prove (4.24), namely,

$$v_j < v_{j+1} \quad (0 \leq j \leq k-3), \quad (4.59)$$

by direct calculations it suffices to show

$$\left(2^{k-j-1} - \frac{j+1}{k+1}\right) w_j < 2^{k-j-1} - \frac{k+1}{2k-1-j} \quad (0 \leq j \leq k-3), \quad (4.60)$$

where w_j is given by

$$w_j = \frac{(2k-2-j)!}{(j+1)!(k+1)^{2k-3-2j}}. \quad (4.61)$$

Moreover, noting that

$$\frac{w_{j+1}}{w_j} = \frac{(k+1)^2}{(j+2)(2k-2-j)} > 1, \quad (4.62)$$

that is, w_j is increasing, we only need to prove

$$\left(2^{k-j-1} - \frac{j+1}{k+1}\right) w_{k-3} < 2^{k-j-1} - \frac{k+1}{2k-1-j} \quad (0 \leq j \leq k-3), \quad (4.63)$$

namely,

$$\left(2^{k-j-1} - \frac{j+1}{k+1}\right) \frac{k(k-1)}{(k+1)^2} < 2^{k-j-1} - \frac{k+1}{2k-1-j} \quad (0 \leq j \leq k-3) \quad (4.64)$$

or

$$\frac{1}{2k-1-j} - \frac{(j+1)k(k-1)}{(k+1)^4} - 2^{k-j-1} \frac{3k+1}{(k+1)^3} < 0 \quad (0 \leq j \leq k-3). \quad (4.65)$$

Denote

$$t = 2k-1-j \quad (k+2 \leq t \leq 2k-1), \quad (4.66)$$

$$g(t) = \frac{1}{t} - \frac{(2k-t)k(k-1)}{(k+1)^4} - 2^{t-k} \frac{3k+1}{(k+1)^3}. \quad (4.67)$$

We next prove

$$g(t) < 0,$$

which leads to our desired result immediately.

In fact, direct calculations show that

$$g'(t) = -\frac{1}{t^2} + \frac{k(k-1)}{(k+1)^4} - 2^{t-k} \frac{3k+1}{(k+1)^3} \ln 2 \quad (4.68)$$

and

$$g''(t) = \frac{2}{t^3} - 2^{t-k} \frac{3k+1}{(k+1)^3} (\ln 2)^2. \quad (4.69)$$

On the other hand, noting

$$g''(k+2) = \frac{2}{(k+2)^3} - \frac{4(3k+1)}{(k+1)^3} (\ln 2)^2 = -\frac{12(\ln 2)^2}{k^2} + O(k^{-3}) < 0, \quad (4.70)$$

$$g'(k+2) = -\frac{1}{(k+2)^2} + \frac{k(k-1)}{(k+1)^4} - \frac{4(3k+1)}{(k+1)^3} \ln 2 = -\frac{12 \ln 2}{k^2} + O(k^{-3}) < 0, \quad (4.71)$$

and using

$$g(k+2) = \frac{1}{k+2} - \frac{k(k-1)(k-2)}{(k+1)^4} - \frac{4(3k+1)}{(k+1)^3} = -\frac{7}{k^2} + O(k^{-3}) < 0, \quad (4.72)$$

we obtain

$$g(t) \leq g(k+2) < 0. \quad (4.73)$$

This proves (4.24).

Finally, we show (4.25).

In fact, noting

$$v_{k-2} - v_{k-1} = u_{k+1} - u_{k-2} - u_k + u_{k-1} = \frac{(k+1)^{k-4}}{(k+3)!} (32k^6 + O(k^5)) 2^{\frac{k^2-k-12}{2}} \quad (4.74)$$

and using

$$v_{k-3} - v_{k-4} = u_{k+2} - u_{k-3} - u_{k+3} + u_{k-4} = \frac{(k+1)^{k-4}}{(k+3)!} (33k^6 + O(k^5)) 2^{\frac{k^2-k-12}{2}}, \quad (4.75)$$

we have

$$v_{k-1} - v_{k-2} + v_{k-3} - v_{k-4} = \frac{(k+1)^{k-4}}{(k+3)!} (k^6 + O(k^5)) 2^{\frac{k^2-k-12}{2}} > 0, \quad (4.76)$$

which leads to the desired (4.25) directly. Thus, the proof of Lemma 4.2 is finished. \blacksquare

In a similar way, we can show the following lemma on more precise results on the sign of function values.

Lemma 4.3 *For large k , it holds that*

$$(-1)^k f\left(-\left(k + \frac{\lambda}{k}\right) 2^{k-1}\right) < 0, \quad \lambda \in \left(0, \frac{66}{25}\right) \quad (4.77)$$

and

$$(-1)^k f\left(-\left(k + \frac{\lambda}{k}\right) 2^{k-1}\right) > 0, \quad \lambda \in \left(\frac{2167}{789}, \infty\right). \quad (4.78)$$

Proof. Since the proof is almost the same as above, we only prove it in a brief way.

For simplicity, we assume $\lambda > 1$. By direct calculations, we have

$$f\left(-\left(k + \frac{\lambda}{k}\right) 2^{k-1}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(k + \frac{\lambda}{k}\right)^n}{n!} 2^{nk-n(n+1)/2} \triangleq \sum_{n=0}^{\infty} (-1)^n u_n \quad (4.79)$$

Similarly, we have

$$u_n \text{ is increasing for } n < k; \quad u_n \text{ is decreasing for } n > k; \quad u_{2k} < 1. \quad (4.80)$$

Denote

$$v_j = u_{2k-1-j} - u_j \quad (0 \leq j \leq k-1). \quad (4.81)$$

By similar calculation and estimation, we get

$$v_j > 0 \quad (0 \leq j \leq k-1) \quad (4.82)$$

and

$$v_j < v_{j+1} \quad (0 \leq j \leq k-3). \quad (4.83)$$

It is easy to see that, for large k

$$v_{k-1} - v_{k-2} + v_{k-3} - v_{k-4} + v_{k-5} = \frac{(k + \frac{\lambda}{k})^{k+2}}{(k+4)!} (25\lambda - 66 + O(k^{-1})) 2^{\frac{k^2-k-20}{2}} \quad (4.84)$$

and

$$v_{k-1} - v_{k-2} + v_{k-3} - v_{k-4} + v_{k-5} - v_{k-6} = \frac{(k + \frac{\lambda}{k})^{k+3}}{(k+5)!} (789\lambda - 2167 + O(k^{-1})) 2^{\frac{k^2-k-30}{2}}. \quad (4.85)$$

Hence, for large k it holds that

$$(-1)^k \sum_{n=0}^{2k-1} (-1)^n u_n = (v_{k-1} - v_{k-2} + v_{k-3} - v_{k-4} + v_{k-5}) - \sum_{j=0}^{k-6} (-1)^{k-j} v_j < 0 \quad \text{for } \lambda < \frac{66}{25} \quad (4.86)$$

and

$$(-1)^k \sum_{n=0}^{2k-1} (-1)^n u_n = (v_{k-1} - v_{k-2} + v_{k-3} - v_{k-4} + v_{k-5} - v_{k-6}) + \sum_{j=0}^{k-7} (-1)^{k-j-1} v_j > 0 \quad \text{for } \lambda > \frac{2167}{789}. \quad (4.87)$$

Similarly, we can show

$$\left| \sum_{n=0}^{2k-1} (-1)^n u_n \right| > 1 \quad (4.88)$$

and

$$\left| \sum_{n=2k}^{\infty} (-1)^n u_n \right| < 1. \quad (4.89)$$

This proves Lemma 4.3. ■

Again, notice

$$v_{k-j} = \frac{(k + \frac{\lambda}{k})^{k+j-3}}{(k+j-1)!} \left[(2j-1)\lambda + \frac{j}{6}(j-1)(2j-1) + O(k^{-1}) \right] 2^{\frac{k^2-k-j(j-1)}{2}}. \quad (4.90)$$

Fixing any integer m with $2m < k$, when k is large enough, we have

$$\sum_{j=1}^{2m-1} (-1)^{j-1} v_{k-j} = \frac{(k + \frac{\lambda}{k})^{k-2}}{k!} \left[\lambda \sum_{j=1}^{2m-1} (2j-1)(-1)^{j-1} 2^{\frac{j-j^2}{2}} + \sum_{j=1}^{2m-1} \frac{j}{6} (j-1)(2j-1)(-1)^{j-1} 2^{\frac{j-j^2}{2}} + O(k^{-1}) \right] 2^{\frac{k^2-k}{2}} \quad (4.91)$$

and

$$\sum_{j=1}^{2m} (-1)^{j-1} v_{k-j} = \frac{(k + \frac{\lambda}{k})^{k-2}}{k!} \left[\lambda \sum_{j=1}^{2m} (2j-1) 2^{\frac{j-j^2}{2}} (-1)^{j-1} + \sum_{j=1}^{2m} \frac{j}{6} (j-1)(2j-1) 2^{\frac{j-j^2}{2}} (-1)^{j-1} + O(k^{-1}) \right] 2^{\frac{k^2-k}{2}}. \quad (4.92)$$

Hence

$$(-1)^k \sum_{n=0}^{2k-1} (-1)^n u_n < 0, \quad \lambda < \frac{\sum_{j=1}^{2m-1} \frac{j}{6} (j-1)(2j-1) 2^{\frac{j-j^2}{2}} (-1)^j}{\sum_{j=1}^{2m-1} (2j-1) 2^{\frac{j-j^2}{2}} (-1)^{j-1}} \quad (4.93)$$

and

$$(-1)^k \sum_{n=0}^{2k-1} (-1)^n u_n > 0, \quad \lambda > \frac{\sum_{j=1}^{2m} \frac{j}{6} (j-1)(2j-1) 2^{\frac{j-j^2}{2}} (-1)^j}{\sum_{j=1}^{2m} (2j-1) 2^{\frac{j-j^2}{2}} (-1)^{j-1}}. \quad (4.94)$$

Moreover, we have

$$\lim_{m \rightarrow \infty} \frac{\sum_{j=1}^{2m} \frac{j}{6} (j-1)(2j-1) 2^{\frac{j-j^2}{2}} (-1)^j}{\sum_{j=1}^{2m} (2j-1) 2^{\frac{j-j^2}{2}} (-1)^{j-1}} = C \approx 2.744. \quad (4.95)$$

This result will be used in the following discussion.

By Proposition 4.1, we can prove

Proposition 4.2 *The solution $f(x)$ has a countably infinite number of different real zeros denoted by $\mathbb{A} = \{x_n\}_{n \geq 1}$ with $x_{n+1} < x_n < 0$. Moreover it holds that*

$$x_n = -(n + \theta_n) 2^{n-1} \quad (0 < \theta_n < 1). \quad (4.96)$$

Proof. Noting that the convergence radius of $f(z)$ in \mathbb{C} is $+\infty$ and $f(z) \not\equiv 0$, we see that $f(z)$ is a nonzero analytic function. Since the zeros of a nonzero analytic function are isolated, we observe that $f(x)$ has an at most countably infinite number of real zeros.

By Proposition 4.1 and intermediate value theorem, $f(x)$ has an infinite number of real zeros. Hence we conclude that $f(x)$ has a countably infinite number of real zeros. Since the real zeros are isolated and negative, the set of real zeros of $f(x)$ can be uniquely represented as $\{x_n\}_{n \geq 1}$ with $x_{n+1} < x_n < 0$.

In what follows, we prove the estimate of x_n .

Notice that $\{2x_n\}_{n \geq 1}$ is the set of critical points of $f(x)$. Then for any integer $k \geq 1$, there are $k - 1$ critical points in the interval $(2x_k, +\infty)$. By mean value theorem, we observe that $f(x)$ has at most k different real zeros and when the number of zeros is k , every zero is simple. Again, by noting

$$\{x_n\}_{1 \leq n \leq k} \subset (2x_k, +\infty), \quad (4.97)$$

we observe immediately that $f(x)$ has exactly k different real zeros in $(2x_k, +\infty)$ and every zero is simple. Hence, we get

$$x_{n+1} < 2x_n \quad (n \geq 1). \quad (4.98)$$

On one hand, for any fixed $x \in [-1, 0)$, it holds that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n! 2^{\frac{n(n-1)}{2}}} = \sum_{n=0}^{\infty} \frac{|x|^n}{n! 2^{\frac{n(n-1)}{2}}} (-1)^n \triangleq \sum_{n=0}^{\infty} a_n (-1)^n. \quad (4.99)$$

Since $\{a_n\}$ is decreasing and

$$a_0 > 0, \quad (4.100)$$

we obtain

$$f(x) > 0, \quad \forall x \in [-1, 0). \quad (4.101)$$

Therefore,

$$x_1 < -1.$$

On the other hand, by Proposition 4.1, $f(x)$ has zeros in interval $(-2, -1)$. Hence

$$x_1 \in (-2, -1). \quad (4.102)$$

In what follows, we finish the proof by induction.

To do so, we assume that

$$x_k \in (-(k+1)2^{k-1}, -k2^{k-1}) \quad (4.103)$$

holds for integer $k \geq 1$. Then

$$x_{k+1} < 2x_k < -k2^k \leq -(k+1)2^{k-1}. \quad (4.104)$$

We next prove

$$x_{k+1} < -(k+1)2^k \quad (4.105)$$

by contradiction.

In fact, assuming

$$x_{k+1} \in (-(k+1)2^k, -(k+1)2^{k-1}), \quad (4.106)$$

then by Proposition 4.1, we have

$$f(-(k+1)2^k)f(-(k+1)2^{k-1}) > 0. \quad (4.107)$$

Since x_{k+1} is simple, there exists a point

$$x^* \in (-(k+1)2^k, x_{k+1}) \quad (4.108)$$

such that

$$f(x^*)f(-(k+1)2^k) < 0. \quad (4.109)$$

Therefore, $f(x)$ has zeros in $(-(k+1)2^k, x^*)$. Hence, we have

$$x_{k+2} \in (-(k+1)2^k, x_{k+1}), \quad (4.110)$$

which contradicts to

$$x_{k+2} < 2x_{k+1} < -(k+1)2^k. \quad (4.111)$$

Consequently, we get

$$x_{k+1} < -(k+1)2^k. \quad (4.112)$$

Since $f(x)$ has zeros in the interval $(-(k+2)2^k, -(k+1)2^k)$, we obtain

$$x_{k+1} \in (-(k+2)2^k, -(k+1)2^k). \quad (4.113)$$

Thus, the desired result has been established for any integer $k \geq 1$. This finishes the proof of Proposition 4.2. \blacksquare

The next proposition is on the truncated polynomials.

Proposition 4.3 *For any given integer $N > 2$, the following truncated polynomial of $f(x)$*

$$P_N(x) = \sum_{n=0}^N \frac{x^n}{n! 2^{\frac{n(n-1)}{2}}} \quad (4.114)$$

has a pair of imaginary zeros $\{z_N, \bar{z}_N\}$ and $N-2$ real zeros, which are denoted by

$$\{r_n\}_{1 \leq n \leq N-2} \quad \text{with} \quad r_{n+1} < r_n < 0. \quad (4.115)$$

Moreover, the following estimates hold

$$|r_n| < (n+2)2^n \quad (1 \leq n \leq N-2) \quad (4.116)$$

and

$$|z_N|^2 > 2^N. \quad (4.117)$$

Proof. The proof consists of six steps. We denote $\xi_k = -(k+1)2^{k-1}$ ($k \geq 0$).

Step I. For $k \in \{1, \dots, \lfloor \frac{N+1}{2} \rfloor\}$, we claim

$$(-1)^k P_N(\xi_k) > 0. \quad (4.118)$$

In fact, by direct calculation, we have

$$P_N(-(k+1)2^{k-1}) = \sum_{n=0}^N (-1)^n \frac{(k+1)^n}{n!} 2^{nk-n(n+1)/2} \triangleq \sum_{n=0}^N (-1)^n u_n. \quad (4.119)$$

Then

$$\frac{u_{n+1}}{u_n} = \frac{k+1}{n+1} 2^{k-n-1}. \quad (4.120)$$

Hence

$$u_n \text{ is increasing for } n < k; \quad u_n \text{ is decreasing for } n > k; \quad u_{2k} < 1. \quad (4.121)$$

As before, we denote

$$v_j = u_{2k-1-j} - u_j, \quad (4.122)$$

and then we obtain

$$v_j > 0 \quad (0 \leq j \leq k-1), \quad (4.123)$$

$$v_j < v_{j+1} \quad (0 \leq j \leq k-3) \quad (4.124)$$

and

$$v_{k-1} - v_{k-2} + v_{k-3} - v_{k-4} > 0. \quad (4.125)$$

Again, we have

$$(-1)^k \sum_{n=0}^{2k-1} (-1)^n u_n > 1. \quad (4.126)$$

Since $2k-1 \leq N$ and

$$\left| \sum_{n=2k}^N (-1)^n u_n \right| < 1, \quad (4.127)$$

we get

$$(-1)^k \sum_{n=0}^N (-1)^n u_n > 0. \quad (4.128)$$

Step II. For any given integer $k = N - 2j$ ($j \in \mathbb{N}_+$) with $2k - 1 > N$, we claim

$$(-1)^N P_N(\xi_k) > 0. \quad (4.129)$$

Similar to Step I, denote

$$v_j = u_{2k-1-j} - u_j. \quad (4.130)$$

Then it holds that

$$v_j > 0 \quad (0 \leq j \leq k-1), \quad (4.131)$$

$$v_j < v_{j+1} \quad (0 \leq j \leq k-3) \quad (4.132)$$

and

$$v_{k-1} - v_{k-2} + v_{k-3} - v_{k-4} > 0. \quad (4.133)$$

Then

$$\begin{aligned} (-1)^N P_N(-(k+1)2^{k-1}) &= \sum_{n=2k-1-N}^{k-1} v_n (-1)^{n-k+1} + \sum_{n=0}^{2k-2-N} u_n (-1)^{N-n} \\ &\geq \sum_{n=2k-1-N}^{k-1} v_n (-1)^{n-k+1} > v_{2k-2-N} > 0. \end{aligned} \quad (4.134)$$

This finishes the proof of Step II.

Step III. For any integer $N > 7$, we claim

$$(-1)^{N-1} P_N(-5N2^{N-6}) > 0. \quad (4.135)$$

In fact, direct calculations yield

$$(-1)^{N-1} P_N(-5N2^{N-6}) = \sum_{n=0}^N (-1)^{n-N+1} \frac{(5N2^{N-6})^n}{n! 2^{\frac{n(n-1)}{2}}} \triangleq \sum_{n=0}^N (-1)^{n-N+1} a_n. \quad (4.136)$$

Then

$$\frac{a_n}{a_{n-1}} = \frac{5N2^{N-6}}{(n+1)2^n}. \quad (4.137)$$

Hence

$$a_n \text{ is increasing for } n < N-3; \quad \text{while } a_n \text{ is decreasing for } n > N-3. \quad (4.138)$$

Notice

$$\begin{aligned}
& (-1)^{N-1} \sum_{n=0}^N (-1)^n a_n \\
&= - \left[(a_N - a_{N-7}) - (a_{N-1} - a_{N-6}) + (a_{N-2} - a_{N-5}) - (a_{N-3} - a_{N-4}) + \sum_{n=N-8}^N (-1)^{n-N} a_n \right], \tag{4.139}
\end{aligned}$$

where

$$a_N - a_{N-7} = \frac{(5N2^{N-6})^{N-7}}{N!2^{\frac{(N-7)(N-8)}{2}}} \left[\frac{61741}{16384} N^7 + O(N^6) \right], \tag{4.140}$$

$$a_{N-1} - a_{N-6} = \frac{(5N2^{N-6})^{N-6}}{N!2^{\frac{(N-6)(N-7)}{2}}} \left[\frac{2101}{1024} N^5 + O(N^4) \right], \tag{4.141}$$

$$a_{N-2} - a_{N-5} = \frac{(5N2^{N-6})^{N-5}}{N!2^{\frac{(N-5)(N-6)}{2}}} \left[\frac{61}{64} N^3 + O(N^2) \right] \tag{4.142}$$

and

$$a_{N-3} - a_{N-4} = \frac{(5N2^{N-6})^{N-3}}{N!2^{\frac{(N-3)(N-4)}{2}}} \left[\frac{1}{4} N + O(1) \right]. \tag{4.143}$$

Direct calculation leads to

$$\begin{aligned}
& (a_N - a_{N-7}) - (a_{N-1} - a_{N-6}) + (a_{N-2} - a_{N-5}) - (a_{N-3} - a_{N-4}) \\
&= \frac{(5N2^{N-6})^{N-7}}{N!2^{\frac{(N-7)(N-8)}{2}}} \left[-\frac{5619}{16384} N^7 + O(N^6) \right] < -\frac{(5N2^{N-6})^{N-8}}{(N-8)!2^{\frac{(N-8)(N-9)}{2}}} = -a_{N-8}. \tag{4.144}
\end{aligned}$$

Therefore, we obtain

$$(-1)^{N-1} \sum_{n=0}^N (-1)^n a_n > 0. \tag{4.145}$$

This proves (4.135).

Step IV. For any given integer $N > 2$, we claim that $P_N(x)$ has imaginary zeros. In what follows, we prove this statement by contradiction.

Assume that $P_N(x)$ only has N real zeros. Then by mean-value theorem, we observe that

$$\frac{d^{N-3}}{dx^{N-3}} P_N(x) \tag{4.146}$$

has 3 real zeros. Direct calculation yields

$$\frac{d^{N-3}}{dx^{N-3}} P_N(x) = 2^{-\frac{(n-3)(n-4)}{2}} + 2^{-\frac{(n-2)(n-3)}{2}} x + \frac{1}{2} 2^{-\frac{(n-1)(n-2)}{2}} x^2 + \frac{1}{6} 2^{-\frac{n(n-1)}{2}} x^3 = 0, \tag{4.147}$$

namely,

$$x^3 + 3 \cdot 2^{n-1} x^2 + 3 \cdot 2^{2n-2} x + 3 \cdot 2^{3n-5} = 0. \tag{4.148}$$

Obviously, the discriminant of the above cubic equation reads

$$\Delta = 2^{6n-12} > 0. \quad (4.149)$$

Therefore, the cubic equation (4.148) has only one real root and a pair of imaginary roots, this contradicts to the assumption.

Step V. For any fixed integer $N > 2$, we claim that the polynomial $P_N(x)$ has $N - 2$ different real zeros denoted by

$$\{x_k\}_{1 \leq k \leq N-2} \quad \text{with} \quad x_k > x_{k+1}, \quad (4.150)$$

and it holds that

$$\xi_{k+1} < x_k < \xi_{k-1} \quad \left(k = N - 2j - 1; \quad j = 1, \dots, \left\lfloor \frac{N-1}{2} \right\rfloor \right) \quad (4.151)$$

and

$$\xi_{N-2} < x_{N-2}. \quad (4.152)$$

It is easy to check that the statement is true for the case that $N = 3, 4, 5, 6, 7$. We next prove this statement by induction. To do so, we firstly assume that the statement holds for integer $N - 2$ ($N > 7$), then we prove it also holds for integer N .

By assumption, the polynomial $P_{N-2}(x)$ has $N - 4$ different real zeros denoted by

$$\{w_k\}_{1 \leq k \leq N-4} \quad \text{with} \quad w_k > w_{k+1}, \quad (4.153)$$

and it holds that

$$\xi_{k+1} < w_k < \xi_{k-1} \quad \left(k = N - 2j - 1; \quad j = 1, \dots, \left\lfloor \frac{N-3}{2} \right\rfloor \right), \quad (4.154)$$

and

$$\xi_{N-4} < w_{N-4}. \quad (4.155)$$

For integer $k = N - 2j - 3$ ($j \in \mathbb{N}_+$) with $2k - 1 > N$, we have

$$|w_k| \leq |w_{N-5}| < |w_{N-4}| < (N - 3)2^{N-5}. \quad (4.156)$$

Hence

$$P_N(w_k)(-1)^{N-1} = \frac{|w_k^{N-1}|}{(N-1)!2^{\frac{(N-1)(N-2)}{2}}} - \frac{|w_k^N|}{N!2^{\frac{N(N-1)}{2}}} = \frac{|w_k^N|}{N!2^{\frac{N(N-1)}{2}}} (N2^{N-1} + w_k) > 0. \quad (4.157)$$

Combining Step II, we find that $P_N(x)$ has zeros in intervals

$$(\xi_{k+1}, w_k) \quad (4.158)$$

and

$$(w_k, \xi_{k-1}), \quad (4.159)$$

where $k = N - 2j - 3$ ($j \in \mathbb{N}_+$) with $2k - 1 > N$. Then it follows from Steps II and III that $P_N(x)$ has zeros in intervals

$$(\xi_{N-2}, -5N2^{N-6}) \quad (4.160)$$

and

$$(-5N2^{N-6}, \xi_{N-4}). \quad (4.161)$$

By Step I, $P_N(x)$ has zeros in intervals

$$(\xi_k, \xi_{k-1}), \quad (4.162)$$

where $k \in \{1, \dots, \lceil \frac{N+1}{2} \rceil\}$. Thus, we may claim that $P_N(x)$ has at least $N - 2$ different real zeros.

In fact, we may divide into two cases to prove it:

Case 1: $N \equiv 0(\text{mod}4)$ or $N \equiv 1(\text{mod}4)$

In the present situation, $P_N(x)$ has zeros in the following intervals: $x_{N-2} \in (\xi_{N-2}, -5N2^{N-6})$, $x_{N-3} \in (-5N2^{N-6}, \xi_{N-4})$, $x_{N-4} \in (\xi_{N-4}, w_{N-5})$, $x_{N-5} \in (w_{N-5}, \xi_{N-6})$, \dots , $x_{\lceil \frac{N+1}{2} \rceil+2} \in (\xi_{\lceil \frac{N+1}{2} \rceil+2}, w_{\lceil \frac{N+1}{2} \rceil+1})$, $x_{\lceil \frac{N+1}{2} \rceil+1} \in (w_{\lceil \frac{N+1}{2} \rceil+1}, \xi_{\lceil \frac{N+1}{2} \rceil})$, $x_{\lceil \frac{N+1}{2} \rceil} \in (\xi_{\lceil \frac{N+1}{2} \rceil}, \xi_{\lceil \frac{N+1}{2} \rceil-1})$, \dots , $x_1 \in (\xi_1, \xi_0)$.

Case 2: $N \equiv 2(\text{mod}4)$ or $N \equiv 3(\text{mod}4)$

In this case, $P_N(x)$ has zeros in the following intervals: $x_{N-2} \in (\xi_{N-2}, -5N2^{N-6})$, $x_{N-3} \in (-5N2^{N-6}, \xi_{N-4})$, $x_{N-4} \in (\xi_{N-4}, w_{N-5})$, $x_{N-5} \in (w_{N-5}, \xi_{N-6})$, \dots , $x_{\lceil \frac{N+1}{2} \rceil+2} \in (w_{\lceil \frac{N+1}{2} \rceil+2}, \xi_{\lceil \frac{N+1}{2} \rceil+1})$, $x_{\lceil \frac{N+1}{2} \rceil+1} \in (\xi_{\lceil \frac{N+1}{2} \rceil+1}, \xi_{\lceil \frac{N+1}{2} \rceil})$, $x_{\lceil \frac{N+1}{2} \rceil} \in (\xi_{\lceil \frac{N+1}{2} \rceil}, \xi_{\lceil \frac{N+1}{2} \rceil-1})$, \dots , $x_1 \in (\xi_1, \xi_0)$.

On the other hand, in Step IV we proved that $P_N(x)$ has at most $N - 2$ real zeros, hence $P_N(x)$ has exactly $N - 2$ different real zeros denoted by $\{x_k\}_{1 \leq k \leq N-2}$ with $x_k > x_{k+1}$. Moreover, the intervals above show that

$$\xi_{k+1} < x_k < \xi_{k-1} \quad \left(k = N - 2j - 1; j = 1, \dots, \left\lceil \frac{N-1}{2} \right\rceil \right) \quad (4.163)$$

and

$$\xi_{N-2} < x_{N-2}. \quad (4.164)$$

Thus, the statement has been established for any integer $N > 2$.

Step VI. Combining Steps I and V gives

$$|x_k| < (k+2)2^k \quad (k = 1, 2, \dots, N-2). \quad (4.165)$$

On the other hand, the relationship between between zeros and coefficients of a polynomial yields

$$|z_N|^2 = \frac{N!2^{\frac{N(N-1)}{2}}}{\left| \prod_{n=1}^{N-2} x_n \right|} > \frac{N!2^{\frac{N(N-1)}{2}}}{\prod_{n=1}^{N-2} |x_n|} > \frac{N!2^{\frac{N(N-1)}{2}}}{\frac{1}{2}N!2^{\frac{(N-1)(N-2)}{2}}} = 2^N. \quad (4.166)$$

Thus, we finish the proof of Proposition 4.3. \blacksquare

By Proposition 4.3, we can prove

Proposition 4.4 *The solution $y = f(z)$ has no imaginary zeros.*

Proof. For any given $R > 0$, on the disc $|z| \leq R$ it holds that

$$\left| \sum_{n=0}^{\infty} \frac{z^n}{n!2^{\frac{n(n-1)}{2}}} \right| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!2^{\frac{n(n-1)}{2}}} \leq \sum_{n=0}^{\infty} \frac{R^n}{n!2^{\frac{n(n-1)}{2}}}. \quad (4.167)$$

This shows that the power series corresponding to the function $f(z)$ is uniformly convergent in any close disc. Obviously, the function $f(z)$ is analytic in \mathbb{C} . On the other hand, for the real number $R > 0$ mentioned above, by Proposition 4.3, on the area

$$\mathbb{D} = \{z : |z| \leq R\} \setminus \{z : \operatorname{Im} z = 0, -R \leq \operatorname{Re} z \leq 0\}$$

$P_N(x)$ has no zeros and does not identically equal to zero obviously for suitably large N .

Suppose that $f(z)$ has an isolated zero α in \mathbb{D} , then there exists a small disc denoted by

$$\Omega = \{z : |z - \alpha| \leq r\} \subset \mathbb{D}$$

such that $f(z)$ is analytic and has no zeros on $\partial\Omega$. Therefore, by the argument principle we obtain

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} dz = 1. \quad (4.168)$$

Nevertheless, for large N it holds that

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{P_N'(z)}{P_N(z)} dz = 0. \quad (4.169)$$

By the uniform convergence in any close disc, we have

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{P_N'(z)}{P_N(z)} dz = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} dz. \quad (4.170)$$

This is a contradiction because of (4.168) and (4.169). Thus we prove that the function $f(z)$ has no zeros in the area \mathbb{D} . Since $R > 0$ is randomly chosen, we finish the proof of Proposition 4.4. \blacksquare

Theorem 3.1 comes from Propositions 4.2 and 4.4 immediately.

Remark 4.1 *By Lemma 4.3, we can obtain a more precise estimate on the parameter θ_n in Theorem 3.1. In fact, it holds that*

$$\frac{66}{25n} < \theta_n < \frac{2167}{789n}$$

for large n . Moreover, we have

$$\lim_{n \rightarrow \infty} n\theta_n = C \approx 2.744. \quad (4.171)$$

We next prove Theorem 3.2.

Proof of Theorem 3.2. Since $\{x_n\}_{n \geq 1}$ is the set of zeros of $f(x)$ and $f(x)$ satisfies

$$f'(x) = f\left(\frac{x}{2}\right), \quad (4.172)$$

$\{2x_n\}_{n \geq 1}$ is the set of critical points of the function $f(x)$. On the other hand, noting that every zero x_n is simple and it holds that

$$x_{n+1} < 2x_n < x_n, \quad (4.173)$$

we have

$$(-1)^n f(2x_n) > 0, \quad (4.174)$$

that is to say, the signs of extrema are alternate.

Direct calculation yields

$$f(2x_k) = f\left(-(k + \theta_k)2^k\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(k + \theta_k)^n}{n!} 2^{nk - n(n-1)/2} \triangleq \sum_{n=0}^{\infty} (-1)^n u_n. \quad (4.175)$$

Then

$$\frac{u_{n+1}}{u_n} = \frac{k + \theta_k}{n+1} 2^{n-k}. \quad (4.176)$$

This implies that

$$u_n \text{ is increasing for } n < k; \quad u_n \text{ is decreasing for } n > k. \quad (4.177)$$

Moreover, noting

$$u_{2k+2} = \frac{(k + \theta_k)^{2k+2}}{(2k+2)!2^{k+1}} < \frac{(k + \theta_k)^{2k+2}}{\left(\frac{2k+2}{e}\right)^{2k+2}2^{k+1}} < \left(\frac{e}{2\sqrt{2}}\right)^{2k+2} < 1, \quad (4.178)$$

we have

$$|f(2x_k)| < \sum_{n=0}^{2k+1} u_n + \left| \sum_{n=2k+2}^{\infty} u_n (-1)^n \right| < (2k+2)u_{k+1} = \frac{2(k+1)^{k+1}}{k!} 2^{\frac{k(k+1)}{2}} + 1 < 2(k+1)e^{k+1} 2^{\frac{k(k+1)}{2}}. \quad (4.179)$$

Hence, for large k it holds that

$$\log_2 |f(2x_k)| < \frac{1}{2}k^2 + 2k. \quad (4.180)$$

Again, noting

$$|\theta_{n+1} - \theta_n| < 1, \quad (4.181)$$

we can choose a smooth function $\theta(t)$ defined on \mathbb{R}_+ such that

$$\theta(n) = \theta_n, \quad 0 < \theta(t) < 1, \quad |\theta'(t)| < 1. \quad (4.182)$$

Since the smooth function $g(t) = -(t + \theta(t))2^t$ is strictly monotonous for $t > 0$, we can make the following transformation

$$x = -(t + \theta(t))2^t \quad (0 < \theta(t) < 1). \quad (4.183)$$

Then

$$\log_2 |x| = t + \log_2(t + \theta(t)). \quad (4.184)$$

Hence for large $|x|$, it holds that

$$\log_2 |f(x)| < \frac{1}{2}t^2 + 2t < \frac{1}{2}(\log_2 |x|)^2 + 2 \log |x|. \quad (4.185)$$

Thus, the proof of Theorem 3.2 is completed. \blacksquare

5 Functional heat equation

This section is devoted to the study on the following Cauchy problem for a nonlocal heat equation

$$\begin{cases} u_t - u_{xx}(t/2, x) = 0 & (t \geq 0, x \in \mathbb{R}), \\ u(0, x) = \varphi(x), \end{cases} \quad (5.1)$$

where $\varphi(x)$ is the initial data.

Theorem 5.1 *Suppose that the initial data $\varphi(x)$ satisfies*

$$\varphi(x) \in L^1(\mathbb{R}) \cap C(\mathbb{R}), \quad (5.2)$$

and its Fourier transform $\hat{\varphi}(\lambda)$ satisfies

$$\hat{\varphi}(\lambda) \in L^1(\mathbb{R}). \quad (5.3)$$

Furthermore, for any fixed time $t \geq 0$, suppose that

$$\hat{\varphi}(\lambda)f(-\lambda^2 t) \in L^1(\mathbb{R}) \quad \text{and} \quad \lambda^2 \hat{\varphi}(\lambda)f(-\lambda^2 t) \in L^1(\mathbb{R}). \quad (5.4)$$

Then there exists a real-value continuous solution $u = u(t, x)$ to the Cauchy problem (5.1). Moreover, assume $\hat{\varphi}(\lambda)$ has no imaginary zeros and $u(t, x) \in L^1(\mathbb{R})$ for any fixed $t \geq 0$, then the Fourier transform of the solution $u = u(t, x)$ has an infinite number of real zeros and no imaginary zeros.

Proof. By making the Fourier transform with respect to the space variable x , the Cauchy problem (5.1) becomes

$$\begin{cases} \frac{d\tilde{u}(t, \lambda)}{dt} = -\lambda^2 \tilde{u}(t/2, \lambda), \\ \tilde{u}(0, \lambda) = \hat{\varphi}(\lambda) \end{cases} \quad (5.5)$$

Direct calculation leads to

$$\tilde{u}(t, \lambda) = \hat{\varphi}(\lambda)f(-\lambda^2 t). \quad (5.6)$$

We now claim the function

$$u(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\varphi}(\lambda)f(-\lambda^2 t)e^{i\lambda x} d\lambda \quad (5.7)$$

is a real-value continuous solution to the nonlocal heat equation.

In fact, since $\varphi(x) \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ and its Fourier transform $\hat{\varphi}(\lambda) \in L^1(\mathbb{R})$, we have

$$u(0, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\varphi}(\lambda)e^{i\lambda x} d\lambda = \varphi(x). \quad (5.8)$$

In the last equality we have made use of the inversion Fourier transformation.

Noting (5.4) and using the dominated convergence theorem yields

$$u_t(t/2, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} -\lambda^2 \hat{\varphi}(\lambda)f(-\lambda^2 t/2)e^{i\lambda x} d\lambda \quad (5.9)$$

and

$$u_{xx}(t/2, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} -\lambda^2 \hat{\varphi}(\lambda)f(-\lambda^2 t/2)e^{i\lambda x} d\lambda. \quad (5.10)$$

Hence, $u = u(t, x)$ solves the equation (1.8). Clearly, $u(t, x)$, $u_t(t, x)$ and $u_{xx}(t, x)$ are continuous.

On the other hand, we claim that $u(t, x)$ is a real-value function.

In fact, the conjugate of $u(t, x)$ is

$$\bar{u}(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(\mu) e^{i\lambda\mu} d\mu f(-\lambda^2 t) e^{-i\lambda x} d\lambda = (\check{\varphi}(\lambda) f(-\lambda^2 t))^\wedge \quad (5.11)$$

where $\hat{\bullet}$ means the Fourier transform of \bullet , while $\check{\bullet}$ means the inverse Fourier transform of \bullet . Since

$$u(t, x) = (\hat{\varphi}(\lambda) f(-\lambda^2 t))^\vee, \quad (5.12)$$

it suffices to prove

$$(\check{\varphi}(\lambda) f(-\lambda^2 t))^\wedge = (\hat{\varphi}(\lambda) f(-\lambda^2 t))^\vee. \quad (5.13)$$

For any fixed $t \geq 0$, let

$$f_n(\lambda) = f(-\lambda^2 t) \chi_{[-n, n]}(\lambda) \quad (n \in \mathbb{N}_+).$$

Clearly, $f_n \in L^1(\mathbb{R})$. Noting that

$$\hat{f}_n = 2\pi \check{f}_n. \quad (5.14)$$

and using the convolution formula and the inversion Fourier formula, we have

$$(\check{\varphi}(\lambda) f_n(\lambda))^\wedge = (\hat{\varphi}(\lambda) f_n(\lambda))^\vee. \quad (5.15)$$

Let $n \rightarrow \infty$ and we obtain the desired equality (5.13) by the dominated convergence theorem. This shows that $u = u(t, x)$ is a real-value continuous solution to the nonlocal heat equation (1.8).

Noting that $u(t, x) \in L^1(\mathbb{R})$ for any fixed $t \geq 0$, we have

$$\int_{-\infty}^{+\infty} u(t, x) e^{-i\lambda x} dx = \hat{\varphi}(\lambda) f(-\lambda^2 t). \quad (5.16)$$

Since we have proven that $f(x)$ has an infinite number of real(negative) zeros and no imaginary zeros, combining this matter and the assumption on $\hat{\varphi}(\lambda)$, we have completed the proof. \blacksquare

6 Conclusions and remarks

In this paper we introduce and investigate a new kind of functional (including ordinary and evolutionary partial) differential equations with the form (1.2). Typical examples of this new kind of functional equations reads, e.g., the equations (1.3)-(1.4) or (1.7)-(1.9), etc. This kind of

new functional differential equations is a new and powerful tool to study some problems arising from both mathematics and physics, more applications in particular to differential geometry and fundamental physics can be expected. In the present paper, we explore new philosophy enjoyed these equations, in particular, for some typical examples, we prove the global existence of smooth solutions, analyze some interesting properties enjoyed by these solutions, and illustrate the differences between this new class of equations and the traditional one. On the other hand, we may replay the variable $t/2$ in (1.2) as t/α ($\alpha > 1$ is a positive constant), and can carry out a similar discussion and obtain similar results.

As the end of this paper, we state the following conjecture:

Conjecture 6.1 *For any $\alpha > 1$, let $y = f_\alpha(x)$ be the solution to the Cauchy problem for the equation $y'(x) = y(\frac{x}{\alpha})$ with the initial data $y(0) = 1$. Let \mathbb{A} be the set of the zeros of $y = f_\alpha(x)$. Then \mathbb{A} is countably infinite and $\mathbb{A} \subset \mathbb{R}$. Moreover, \mathbb{A} can be denoted by $\{x_n\}_{n \geq 1}$ with $x_n > x_{n+1}$, and it holds that*

$$x_n = -(n + \theta_n)\alpha^{n-1}, \quad (6.1)$$

where $\theta_n \in (0, M_\alpha)$ is a parameter depending on n and α , in which M_α is a positive constant only depending on α but independent of n . Moreover, $\{\theta_n\}$ satisfies

$$\lim_{n \rightarrow \infty} n\theta_n = \frac{\alpha}{3}(\ln h(\alpha))', \quad (6.2)$$

where $h(\alpha)$ is defined by

$$h(\alpha) = \sum_{k=1}^{\infty} (2k-1)(-1)^{k-1} \alpha^{\frac{k-k^2}{2}}. \quad (6.3)$$

Remark 6.1 *In fact, in a similar manner, we can easily prove the conjecture for the case $\alpha \geq 2$. For the case $\alpha \in (1, 2)$, it is worthy to study in future.*

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